The Fine Structure of Nuclear Energy Levels on the Alpha-Model*

CHARLES KITTEL**
Department of Physics, University of Wisconsin, Madison, Wisconsin
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In studying the fine structure predictions of any proposed theory of nuclear forces it is important to know to what extent the results obtained are sensitive to the changes in the nuclear model employed in making the calculations. The present work is concerned with this. The splitting for He^5, which should be very similar to that for Li^5, is discussed on the alpha- and central field models. A deuteron group model for the alpha-particle is considered in addition to the simple alpha-group. Li^7 is treated from the standpoint of the theory of "holes" and also on a modified alpha-particle model. The contribution of the exchange terms to the splitting in He^5 is greater by a factor of the order of two or three on the alpha-model than on the central field model. The sign and magnitude of the total splitting are of about the same order on the two models. The deuteron group alpha-function gives essentially the same splitting as the ordinary alpha-function. So far as ordinary terms for Li^7 are concerned, the spin-orbit interaction of a "hole" in a concentrated closed shell with a closed shell is the same, apart from a numerical factor, as the spin-orbit interaction of a single particle with a closed shell. The exchange terms for Li^7 are approximately proportional to the degree of overlapping of the alpha- and triton groups. The exchange splitting has the same sign as the ordinary splitting. Energy splitting estimates on the alpha-model in which the exchange terms are neglected should give results reliable to an order of magnitude at least, as far as this model is concerned.

I. INTRODUCTION

VARIOUS experiments suggest that the lowest states of the nuclei He^5, Li^5, and Li^7 are doublets. Calculations of Feenberg and Wigner[1] indicate that the ground states of these nuclei should be ^2P. It is logical to interpret the doublets as consisting of the $^2P_{3/2}$ and $^2P_{1/2}$ parts of a ^2P state.

The evidence relating to He^5 comes from the experiments of Staub and Tatel[2] on the scattering of neutrons by helium. The variation with neutron energy of the backward scattering cross section indicates a doublet structure with a splitting of about 0.3 Mev. The sign of the splitting is not fixed.

Experiments by Heydenburg and Ramsey[3] on the scattering of protons by helium show a very broad maximum in the scattering of half-width over 1 Mev. There is no positive evidence of a doublet structure for Li^5.

In the case of Li^7 the ground state is stable, and its spin is[4-7] 3/2. Rumbaugh and Hafstad[8] found an excited state of this nucleus about 400 kev above the normal level.

Theoretical investigations of the fine structure of these nuclei can, as pointed out by Breit and Stehn,[9] throw light on the form of the interaction energy between pairs of nuclear particles. Also, since the fine structure matrix elements involve the derivative of the potential and hence may magnify the influence of the shape of some potentials at small interparticle separations, studies of fine structure offer a possible way of investigating the potential at small distances. It is difficult to obtain this knowledge from scattering experiments on elementary particles, since, according to Hoisington, Share, and Breit,[10] scattering results depend mostly on the shape of the outer region of the potential well.

In testing the fine structure predictions of any proposed theory of nuclear forces it is important to know to what extent the results obtained are sensitive to the changes in the nuclear model employed in making the calculations. The splitting for He^5, which on present ideas should be

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** Now at Naval Ordnance Laboratory, Washington, D. C.

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very similar to that for Li^5, will be discussed on the alpha-particle and central field models. Li^7 is treated on a modified alpha-model, the central field model having been discussed previously by Breit and Stehn.[9]

Dancoff[11] has given numerical estimates for the splitting in He^5 on a central field model, using the relativistic spin-orbit coupling and the tensor spin-orbit-spin coupling of meson theory. Inglis[12] has made estimates of the spin-orbit coupling on the alpha-model of light nuclei, including Li^7.

The theoretical basis for the alpha-particle model has been thoroughly discussed by Wheeler[13] in his papers on the resonating group model. For He^5 the relatively large binding energy of an alpha-particle suggests that the most important grouping of particles will be He^4+n^1, while for Li^7 it will be He^4+H^3. The internal motion of the particles in each group is described by a wave function depending on the interparticle distances; the relative motion of the groups is described by a function depending on the vector from the center of mass (c.m.) of one group to the c.m. of the other; the total wave function is the product of these functions.

To simplify the treatment of Li^7 the alpha-model is modified by introducing central field functions for the particles in the alpha-group and for the center of mass of the triton (H^3) group. On the basis of this model it will be shown that the contribution to the splitting from the exchange terms is approximately proportional to the degree of overlapping of the two groups, for small overlapping. This result should have application in approximate discussions of heavier nuclei.

Breit[14] has shown that the requirements of relativistic covariance of the wave equation for nuclear particles determines the spin-orbit interactions apart from certain adjustable parameters. From a consideration of the possible forms of wave equations for particles with spin he obtains the interaction energy

$$H=\sum \mathbf{B}_{kl}[a\boldsymbol{\sigma}_k+(a-1)\boldsymbol{\sigma}_l]-\sum{}' J_{kl}, \qquad (1)$$

$$\mathbf{B}_{kl}=-(\hbar/4M^2c^2)[\mathbf{p}_k\times\nabla_k J_{kl}], \qquad (2)$$

for Wigner forces. The particles in the nucleus are referred to by subscripts k, l; r_{kl} is the distance between particles k and l; the potential energy between two particles is $-J(r_{kl})$; the momentum operator is written $\mathbf{p}_k=(\hbar/i)\nabla$. $\sum'$ refers to sums over pairs of particles, and $\sum$ to sums over all values of k and l except $k=1$. The constant a is arbitrary. For particles interacting through the electromagnetic field the value of a is -1. The picture of each particle producing a scalar field defined in the reference system moving instantaneously with it corresponds to $a=1$. In the present work only $a=1$ is considered.

Note on Notation

Ψ^A is written for a complete antisymmetrized wave function, including spin; Ψ is the unsymmetrized product wave function, including spin; ψ is the space part of Ψ.

R, Q, $\mathfrak{R}$, ρ have different meanings on different models and must be watched carefully.

P_{ij} is the permutation operator, and interchanges i and j in the expression to its right.

$S_0(12)=(\alpha_1\beta_2-\alpha_2\beta_1)/\sqrt{2}$, where α and β are the usual spin functions. α is for spin up, β for spin down. Indices 1, 2, refer to the two particles.

$$\mathbf{1}=\mathbf{r}_1,\text{ etc.};\quad x_{ij}=x_i-x_j.$$

II. FINE STRUCTURE OF He^5

The resonating group wave function assumed here for the $p_{3/2}$ state of He^5 is

$$\Psi^A=(1-P_{45}-P_{35})v[\mathbf{5}-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4]$$
$$\times u(r_{12},r_{13},r_{14},r_{23},r_{24},r_{34})S_0(12)S_0(34)\alpha(5). \qquad (3)$$

Particles 1 and 2 are protons; 3, 4, and 5 are neutrons. u describes the internal motion of the alpha-particle group and is an s function: unless specifically mentioned to the contrary u will be assumed symmetric in all particles in the alpha-group. The function v describes the motion of the outer neutron relative to the center of mass of the alpha-group. v is a p function as is explained by its dependence on x, y, z in Eq. (4).

Introducing

$$\boldsymbol{\rho}=\mathbf{5}-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4$$
$$=(x_\rho, y_\rho, z_\rho)$$

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in Cartesian coordinates, one writes

$$v(\boldsymbol{\rho})=(x_\rho+iy_\rho)R(\rho), \tag{4}$$

where R is spherically symmetric.

The first-order perturbation energy for the $p_{3/2}$ state is

$$(\Delta E)_{3/2}=(\Psi^A,\, H'\Psi^A)/(\Psi^A,\, \Psi^A). \tag{5}$$

Neglecting the contribution from exchange terms, one finds

$$(\Psi^A,\, H'\Psi)_{\text{ordinary}}=(\lambda^2/4)\int u^2(1234)R^2(\rho)\times(2/3)(\boldsymbol{\rho}\cdot\nabla_5 J_{51})d\tau, \tag{6}$$

a form which shows clearly the dependence on the radial component of the force on the outside particle. A more convenient form is obtained by integrating the original equation by parts.

The total ΔE is three times the displacement of the $p_{3/2}$ state:

$$(\Delta E)_{\text{ordinary}}=-[3\lambda^2\int u^2(1234)J_{51}[2R^2(\rho)+(4/3)R(\rho)\rho R'(\rho)]d\tau]/[(2/3)\int u^2(1234)\rho^2R^2(\rho)d\tau], \tag{7}$$

where only the ordinary term has been included in the numerator as well as in the denominator.

Evaluation of $(\Delta E)_{\text{ordinary}}$ for Gaussian Functions

Let

$$J=B\exp[-br^2], \tag{8a}$$

$$u(1234)=\exp[-\alpha(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)], \tag{8b}$$

$$R(\rho)=\exp[-\beta\rho^2], \tag{8c}$$

where B, b, α, and β are arbitrary constants.

The following general form of integral is needed:

$$\begin{aligned}\int\exp[&-c_1r_1^2-c_2r_2^2-c_3r_3^2-c_4r_4^2-d_1r_{12}^2-d_2r_{13}^2\\&-d_3r_{14}^2-d_4r_{23}^2-d_5r_{24}^2-d_6r_{34}^2]d\tau_{1234}\\&=\pi^6[c_1+c_2+c_3+c_4]^{-\frac{3}{2}}[(a+b+c)(a_3a_1+a_1a_2+a_2a_3)\\&\quad+b(a_3c+a_1a)+c(a_1a+a_2b)+a(a_3c+a_2b)+abc]^{-\frac{3}{2}},\end{aligned} \tag{9}$$

where

$a_1=[(c_2c_3)/(c_1+c_2+c_3+c_4)]+d_4$;
$a_2=[(c_1c_3)/(c_1+c_2+c_3+c_4)]+d_2$;
$a_3=[(c_1c_2)/(c_1+\cdots)]+d_1$; $a=[(c_1c_4)/(c_1+\cdots)]+d_3$;
$b=[(c_2c_4)/(c_1+\cdots)]+d_5$; $c=[(c_3c_4)/(c_1+\cdots)]+d_6$.

This integral was evaluated with the help of the standard form

$$\int\exp[-px^2\pm qx]dx=(\pi/p)^{\frac{1}{2}}\exp[q^2/4p].$$

The integrand in (9) is separable into the form $f(x)f(y)f(z)$, so that it is sufficient to evaluate $\int f(x)dx_1dx_2dx_3dx_4$ and then take the cube of the result.

The integral can also be expressed as (see Fig. 1)

$$\pi^6[\text{all quadruplets of lines except those forming triangles and quadrilaterals}]^{-\frac{3}{2}}. \tag{10}$$

There are 125 terms in the square brackets. Other integrals used in this work are obtained by differentiation with respect to the constants in the integrand.

One obtains

$$(\Delta E)_{\text{ordinary}}=-6\lambda^2Bb[(32\alpha\beta)/(16\alpha b+(32\alpha+3b)\beta)]^{5/2}. \tag{11}$$

Exchange Terms on the Alpha-Particle Model

The exchange integral in (5) simplifies to

$$\int v^*(\boldsymbol{\rho})u(1234)[2B_{34}{}^z-6B_{41}{}^z-2B_{45}{}^z]v(\boldsymbol{\xi})u(1235)d\tau, \tag{12}$$

under the special assumption that u is of the form

$$\exp[-\alpha(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)]. \tag{13}$$

The evaluation of these integrals with Gaussian wave functions and potentials is exceedingly lengthy, and the results difficult to interpret. It is more instructive to consider the values of the integrals in the case of a long neutron range. This has some bearing on the physical situation, because the forces between the alpha-particle and the incoming neutron in a scattering experiment are thought to be relatively weak.

In the "long neutron range approximation" [15] it is assumed that $\alpha\gg\beta$. Particles 1, 2, 3 (in the unsymmetrized wave function) are assumed to be concentrated in a tight group. In what follows the "spread" of 1, 2, 3, will be neglected. The angular factors in the integral are considerably simplified under these conditions, and the integral over **1**, **2**, **3** may be replaced by an integral over **1** alone. One obtains a long formula for $(\Delta E)_{\text{exchange}}$ which the author will be glad to furnish to those interested.

Numerical results are given in Tables I–IV. The values of the constants used are, in nuclear units:

Formula	Constant	Value	Range in 10^{-13} cm
$J=B\exp(-br^2)$	b	20	$2.01=10^{+13}/\sqrt{b}$
$u=\exp[-\alpha\Sigma' r_{ij}^2]$	α	6.1	$3.64=10^{13}/\sqrt{\alpha}$
$R(\rho)=\exp(-\beta\rho^2)$	β	0.81	$10.0=10^{13}/\sqrt{\beta}$
		1.65	7.0
		5.06	4.0

The value of b chosen is in fair accord with the interpretation of experiments on the scattering of protons by protons. A variational calculation to be described later is the basis for the value of α. In an expression of the form $\exp[-cr^2]$, the range is defined as $1/\sqrt{c}$.

From Table I it is seen that the "long neutron range" [l.n.r.] approximation is reliable for the range of values for β considered. Table II gives the contribution to ΔE due to the exchange terms. Table III gives the contribution to ΔE from the ordinary terms, but differs from Table I in that exchange terms are included in the normalization integral now. Table IV gives the ratio of exchange terms to ordinary terms, with

[15] Referred to as l.n.r. approximation.

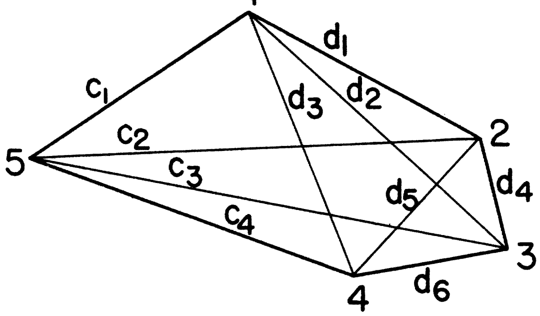

FIG. 1.

the normalization integral including exchange terms in both cases. It is seen from this table that the exchange terms give a very appreciable addition to the splitting on the alpha-particle model.

Central Field Model

Certain results obtained by Breit and Stehn in their treatment of the fine structure of the nuclear ground level of Li^7 can be used in the discussion of He^5 on the central field model. The wave function is

$$\Psi^A=(1/\sqrt{3})[1-P_{45}-P_{35}]\times[\xi_5Q_5R_3R_4S_0(34)\alpha(5)]R_1R_2S_0(12). \quad (15)$$

Here $\xi=x+iy$, Q describes the particle outside the alpha-shell, and R a particle in the alpha-shell. Suppose

$$J=A\exp[-ar^2],\quad Q=N_Q\exp[-\nu r^2/2],\quad R=N_R\exp[-\mu r^2/2].$$

Introduce the abbreviations:

$$d^2=(\mu+\nu)^2/4+a(\mu+\nu),\quad g^2=\mu\nu+a(\mu+\nu).$$

One has

$$\Delta E=[(\Delta E)^{w1}_{s\pi}-(\Delta E)^{w1}_{s\nu}+2(\Delta E)^{w1}_{s\nu}, \quad (16)$$

where the expressions on the right side are defined and evaluated in the work of Breit and Stehn. From this

$$\Delta E=-(3/2)\lambda\!\!\!^{-2}Aa\nu^{5/2}\mu^{5/2}[4g^{-5}+d^{-5}]. \quad (17)$$

For $\mu=\nu$, it follows that $d=g$, and

$$\Delta E=-(15/2)\lambda\!\!\!^{-2}Aa\nu^{5/2}\mu^{5/2}g^{-5}. \quad (18)$$

In this case the exchange term is 0.20 of the total ΔE.

As a preliminary to the comparison of the fine structure as calculated on different models, it is useful to have relations between the Gaussian function $u(1234)=\exp[-\alpha(r_{12}^2+\cdots+r_{34}^2)]$ in the interparticle distances, and the product function

$$u(1234)=\exp[-\mu r_{a1}^2/2]\exp[-\mu r_{a2}^2/2]\times\exp[-\mu r_{a3}^2/2]\exp[-\mu r_{a4}^2/2], \quad (19)$$

where $\mu/2$ is written for comparison with the central field function $R_1=N_R\exp[-\mu r_1^2/2]$, although in the present case r_{a1} differs from r_1, the former being taken relative to the center of mass of the alpha-particle group:

$$\mathbf{r}_{a1}=\mathbf{1}-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4,$$
$$\mathbf{r}_{a2}=\mathbf{2}-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4, \quad (20)$$
$$\cdots\cdots$$

If we let $\alpha=\mu/8$, $\beta=(8\nu+2\mu)/25$, or

$$\mu=8\alpha,\quad \nu=25\beta/8-2\alpha, \quad (21)$$

the product function in the five particles is found equal to the expression in r_{12}, $\cdots$, r_{34}, ρ.

Central field functions are referred to a fixed point—therefore the above relations are not strictly applicable. They will be used, however, in estimating the μ and ν to be used for comparison with a given α, β. This amounts to neglecting the center of mass motion on the central field model.

For $\mu=\nu$, $\alpha=\mu/8$; $\beta=2\mu/5$, or $\beta=16\alpha/5$. With $\alpha=6.1$, $\beta=19.52$ (range $\sim2\times10^{-13}$ cm). This value of β is unlikely physically, but it is apparent from Table IV that on the alpha-particle model the exchange terms are relatively more important than on the central field model.

Using (21) one has for $(\Delta E)_{\text{ordinary}}$ on the central field model,

$$(\Delta E)_{\text{ordinary}}=-6\lambda\!\!\!^{-2}Bb[(25\alpha\beta-16\alpha^2)/(25\alpha\beta-16\alpha^2+b(25\beta/8+6\alpha))]^{5/2}. \quad (22)$$

TABLE I. Table of values of $(\Delta E)_{\text{ordinary}}/B$. Only ordinary terms included in normalization integral.

β	Exact	l.n.r. approx.
0.81	-9.48×10^{-5}	-9.50×10^{-5}
1.65	-4.46×10^{-4}	-4.50×10^{-4}
5.06	-3.33×10^{-3}	-3.41×10^{-3}

Geometrical Conditions for Vanishing of Exchange Terms on Alpha-Particle Model

It can be shown that on the alpha-group model of He^5 non-vanishing exchange integrals exist even when the alpha-group does not overlap the outside neutron. It is in fact necessary that r_b, the radius of the sphere about the center of mass of the alpha-particle from which the outside neutron is excluded, be greater than $5r_a/3$, where r_a is the radius of the alpha-particle, in order that a non-vanishing exchange integral exist. The reason for this behavior is that the center of mass of the original particles in the alpha-group is displaced when the outside neutron is exchanged with a neutron in the alpha-group.

III. DEUTERON GROUP MODEL OF THE ALPHA-PARTICLE

Up to now the alpha-particle wave function has been used in the form

$$u(1234)=\exp[-\alpha(r_{12}^2+\cdots+r_{34}^2)].$$

It is of interest to consider the effect on the level splitting of a change on the form of u. The alpha-group can be thought of as composed of two deuteron groups; u will be represented as the product of factors representing the internal motion of each deuteron with a factor describing the relative motion of the center of masses of the deuterons. The complete alpha-particle wave function is assumed to be

$$\begin{aligned}\Psi^A=\{&\exp[-\alpha_{\rm I}(r_{12}^2+r_{34}^2+r_{14}^2+r_{23}^2)\\&-\alpha_{\rm II}(r_{13}^2+r_{24}^2)]\\&+\exp[-\alpha_{\rm I}(r_{12}^2+r_{34}^2+r_{13}^2+r_{24}^2)\\&-\alpha_{\rm II}(r_{14}^2+r_{23}^2)]\}S_0(12)S_0(34).\end{aligned}\tag{23}$$

The relationship of this wave function to a deuteron grouping is seen from the relation

$$\begin{aligned}\exp[&-\alpha_{\rm I}(r_{12}^2+r_{34}^2+r_{14}^2+r_{23}^2)\\&-\alpha_{\rm II}(r_{13}^2+r_{24}^2)]=\exp[-\alpha_{\rm I}r_D^2\\&-(\alpha_{\rm I}+\alpha_{\rm II})(r_{13}^2+r_{24}^2)],\end{aligned}\tag{24}$$

where $\mathbf{r}_D=\mathbf{1}+\mathbf{3}-\mathbf{2}-\mathbf{4}$, or twice the vector between the centers of mass of the two deuteron groups. This follows from

$$r_D^2=r_{12}^2+r_{14}^2+r_{23}^2+r_{34}^2-r_{13}^2-r_{24}^2.\tag{25}$$

Two terms in the spatial part of Ψ^A are necessary to make the space part symmetric in 1, 2 and 3, 4.

The ranges $\alpha_{\rm I}$ and $\alpha_{\rm II}$ will be determined by minimizing the energy integral

$$E=(\Psi^A, H\Psi^A)/(\Psi^A, \Psi^A),\tag{26}$$

$$H=T+V,\tag{27}$$

$$T=-\frac{\hbar^2}{M}\left[\sum{}'\left(\frac{\partial^2}{\partial r_{kl}^2}+\frac{2}{r_{kl}}\frac{\partial}{\partial r_{kl}}\right)+\sum_{\substack{l,\,n\neq k\\ l<n}}^{4}\sum_{k=1}^{4}\left(\cos(lkn)\frac{\partial^2}{\partial r_{kl}\partial r_{kn}}\right)\right],\tag{28}$$

$$V=-\sum{}'\,[(1-g)P^M+gP^H]J_{ij};\quad J=A\exp[-br^2],\tag{29}$$

where P^M is the Majorana exchange operator, and P^H the Heisenberg exchange operator. The effect of g is to decrease the attraction between like particles relative to that between unlike particles.

The minimization of E is done by first minimizing the symmetric function with $\alpha_{\rm I}=\alpha_{\rm II}$. With the value of $\alpha_{\rm I}$ found in this manner, $\alpha_{\rm II}$ is varied in the general expression for E, and the minimum determined graphically. It is too laborious to minimize E simultaneously with respect to both $\alpha_{\rm I}$ and $\alpha_{\rm II}$. The results are given in

TABLE II. Table of values of $(\Delta E)_{\rm exchange}/B$.

β	l.n.r. approx.
0.81	-4.50×10^{-5}
1.65	-2.47×10^{-4}
5.06	-2.48×10^{-3}

TABLE III. Table of values of $(\Delta E)_{\rm ordinary}/B$. Exchange terms included in normalization integral.

β	l.n.r. approx.
0.81	-9.48×10^{-5}
1.65	-4.43×10^{-4}
5.06	-2.94×10^{-3}

TABLE IV: Exchange terms included in normalization integral.

β	$[(\Delta E)_{\rm exchange}/(\Delta E)_{\rm ordinary}]$ l.n.r. approx.
0.81	0.475
1.65	0.557
5.06	0.843

Table V, in which we use as constants $A=85.0$, $b=20.0$, $g=0.22$. Nuclear units are used. For $\alpha_{\rm I}=\alpha_{\rm II}$, the minimum occurs at $\alpha_{\rm I}=6.10$, where $E=-52.54\ mc^2$. With this value of $\alpha_{\rm I}$, a graph of E against $\alpha_{\rm II}$ indicates a minimum at $\alpha_{\rm II}=6.5\pm0.05$. The effect of g is to increase the internal binding of the deuteron subgroups.

An alternate form of deuteron grouping was tried in which the spin terms are built up of deuteron spin functions:

$$\Psi_1{}^A=\exp\left[-\alpha_{\rm I}(r_{12}{}^2+r_{34}{}^2+r_{14}{}^2+r_{23}{}^2)-\alpha_{\rm II}(r_{13}{}^2+r_{24}{}^2)\right]\mathcal{S}_1 -\exp\left[-\alpha_{\rm I}(r_{12}{}^2+r_{13}{}^2+r_{34}{}^2+r_{23}{}^2) -\alpha_{\rm II}(r_{14}{}^2+r_{23}{}^2)\right]\mathcal{S}_2; \quad (30)$$

$$\mathcal{S}_1=(1/\sqrt{3})(S_1(13)S_{-1}(24)-S_0(13)S_0(24) +S_{-1}(13)S_1(24)), \quad (31)$$

$$\mathcal{S}_2=(1/\sqrt{3})(S_1(14)S_{-1}(23)-S_0(14)S_0(23) +S_{-1}(14)S_1(23)). \quad (32)$$

For $\alpha_{\rm I}=\alpha_{\rm II}$, this function becomes identical with the one previously considered. The binding energy is slightly less with the present function. It can be shown that $\Psi_1{}^A$ differs from Ψ^A only by the addition of a small term with like particle spins coupled parallel. Since Heisenberg forces favor antiparallel coupling, $\Psi_1{}^A$ is not as good a function as Ψ^A. The fine structure was not calculated for $\Psi_1{}^A$.

Estimates of $(\Delta E)_{\rm ordinary}$ on the Different Models

Constants used: $b=20$, $\alpha=\alpha_{\rm I}=6.10$, $\alpha_{\rm II}=6.60$, $B=75$. Equation (22) used for central model.

$\beta=10$

$$(\Delta E)_{\rm alpha}=-6\lambda^2Bb[0.1237]=-0.605\ mc^2,$$
$$(\Delta E)_{\rm alpha\text{-}deuteron}=-6\lambda^2Bb[0.1248]=-0.612\ mc^2,$$
$$(\Delta E)_{\rm central}=-6\lambda^2Bb[0.1054]=-0.516\ mc^2.$$

$\beta=19.52$ (when $\mu=\nu$)

$$(\Delta E)_{\rm alpha}=(\Delta E)_{\rm central} =-6\lambda^2Bb[0.224]=-1.1\ mc^2.$$

Concentrated Alpha-Particle ($\alpha\rightarrow\infty$)

$$(\Delta E)_{\rm alpha}=-6\lambda^2Bb[2\beta/(b+2\beta)]^{5/2} =-6\lambda^2Bb[0.176]\quad\text{for}\quad\beta=10 =-6\lambda^2Bb[0.36]\quad\text{for}\quad\beta=20.$$

In this limit the exchange terms vanish, and the above values give the total splitting.

Central Model, letting $\mu\rightarrow\infty$

$$(\Delta E)_{\rm central}=-6\lambda^2Bb[\nu/(\nu+b)]^{5/2}.$$

It is seen from these results that so far as the ordinary terms are concerned the "deuteron-grouping" alpha-function gives essentially the same ΔE as the simple alpha-function

$$u=\exp\left[-\alpha(r_{12}{}^2+\cdots+r_{34}{}^2)\right].$$

TABLE V. $\alpha_{\rm I}=6.10$. Energy in mc^2.

$\alpha_{\rm II}=$	5.0	6.1	6.2	6.6	7.0
Kinetic energy	103.1	109.80	110.32	112.631	114.961
Potential energy	−155.07	−162.34	−162.86	−165.244	−167.296
Total energy	−52.0	−52.54	−52.54	−52.61	−52.34

The results are not very different from those calculated on the central model with corresponding ranges. In comparing with the central model no correction has been made for the center of mass motion.

IV. SPLITTING DUE TO INTERACTION OF A CONCENTRATED s GROUP WITH AN ALPHA-GROUP

It will be shown that the spin-orbit interaction of an s group, in an $s_{1/2}$ state of internal motion, with an alpha-group is the same apart from a numerical factor as the interaction of a single particle with an alpha-group, in the limit of a very concentrated space function for the outside group. This calculation is of interest in connection with the theory of holes. The result will be given for the case of a $p_{3/2}$ state of the complete system.

Let N be the total number of particles, of which $N-4$ are in a group in an $s_{1/2}$ state of internal motion, and 4 are in an alpha-group. The exchange terms vanish in the limit of a concentrated function for the outside group; thus ordinary terms only will be considered. The result will be given for any odd integer N in order to exhibit the numerical constant mentioned above; in practice only $N=5$ and 7 are possible because of the exclusion principle.

The wave function is

$$\Psi=u(r_{12},\cdots,r_{34})w(r_{56},\cdots,r_{N-1,\,N})v(\boldsymbol{\varrho})S_0(12) \times S_0(34)S_0(56)\cdots S_0(N-2,\,N-1)\alpha(N);$$
$$v(\boldsymbol{\varrho})=(x_\rho+iy_\rho)R(|\rho|),$$

where

$$\boldsymbol{\varrho}=(\mathbf{5}+\mathbf{6}+\cdots+\mathbf{N})/(N-4)-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4. \quad (33)$$

One finds

$$(\Psi,H'\Psi)=-\lambda^2[N-4]^{-2}\int J_{N1}u(1234)w^2(5\cdots N) \times[2R^2(\rho)+4/3\rho R'(\rho)R(\rho)]d\tau_{1\ldots N}. \quad (34)$$

To see the resemblance to the corresponding result for He^5, one introduces a delta-function for w:

$$w(5\cdots N)=[\delta(\mathbf{N}-\mathbf{5})\delta(\mathbf{N}-\mathbf{6})\delta(\mathbf{N}-\mathbf{7})\cdots \delta(\mathbf{N}-[\mathbf{N}-\mathbf{1}])]^{\frac{1}{2}}, \quad (35)$$

where $\delta(\mathbf{N}-\mathbf{5})$ denotes $\delta(x_N-x_5)\delta(y_N-y_5)\delta(z_N-z_5)$. Then the integral becomes

$$\int J_{N1}u^2(1234)[2R^2(\bar{\rho})+4/3\bar{\rho}R'(\bar{\rho})R(\bar{\rho})]d\tau_{1234N},$$

where $\bar{\boldsymbol{\varrho}}=\mathbf{N}-(\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4})/4$. Finally

$$(\Delta E)=-3\lambda^2\left[\int u^2(1234)J_{N1}[2R^2(\bar{\rho}) +4/3\rho R'(\bar{\rho})R(\bar{\rho})]d\tau_{1234N}\right]/ \left[[N-4]^2 2/3\int u^2(1234)\bar{\rho}^2R^2(\bar{\rho})d\tau_{1234N}\right] =[N-4]^{-2}(\Delta E)_{\text{ordinary, alpha-model, He}^5}. \quad (36)$$

The factor $[N-4]^{-2}$ can be understood from a classical model, according to which

$$\Delta E/\hbar=\omega=-\mathbf{v}\times\mathbf{a}/2c^2=-\dot{r}_\perp\ddot{r}_{\mathrm{radial}}/2c^2. \quad (37)$$

For a given angular momentum $\dot{r}_\perp$ is $[N-4]^{-1}$ that for He^5, and $\ddot{r}_{\mathrm{radial}}$ is $[N-4]^{-1}$ that for He^5, so that $[N-4]^{-2}$ enters the result.

The preceding results are also applicable to a model in which central field product functions are used for the alpha-particle, and also for the center of mass of the outside group. In this case one finds

$$(\Delta E)=[N-4]^{-2}(\Delta E)_{\text{ordinary, central model, He}^5}. \quad (38)$$

V. FINE STRUCTURE OF Li^7

The nucleus Li^7 is pictured on the alpha-particle model as built up from an alpha-particle group and a triton (H^3) group. The unsymmetrized wave function for the $p_{3/2}$ state is taken to be of the form

$$\Psi=u(1234)w(r_{56},r_{57},r_{67})v(\boldsymbol{\varrho})S_0(12)\times S_0(34)S_0(56)\alpha(7), \quad (39)$$

so that the antisymmetric wave function is

$$\Psi^A=(1-P_{17}-P_{27})(1-P_{35}-P_{36}-P_{45}-P_{46}+P_{35}P_{46})\Psi. \quad (40)$$

ψ will be used to denote the space part of Ψ. Particles 1, 2, 7 are protons; 3, 4, 5, 6 are neutrons; u is a central field product type function,

$$u(1234)=\Re(1)\Re(2)\Re(3)\Re(4), \quad (41)$$

$$v(\boldsymbol{\varrho})=(x_\rho+iy_\rho)Q(\rho), \quad (42)$$

where

$$\boldsymbol{\varrho}=\tfrac{1}{3}(\mathbf{5}+\mathbf{6}+\mathbf{7}).$$

This is a central field representation of $v(\boldsymbol{\varrho})$. w is an s function, and will be assumed symmetric in 5, 6, 7.

$$(\Psi^A,H'\Psi^A)=18(\Psi,H'\Psi^A). \quad (43)$$

$$(\Psi,H'\Psi^A)=I_0+I_1+I_2+I_3, \quad (44)$$

where I_0 contains the terms classified as ordinary, I_1 the terms with a single permutation operator, I_2 the terms with two permutation operators, and I_3 those with three permutation operators.

The ordinary terms can be evaluated from the equation for central field functions analogous to (34), with $N=7$. Setting

$$\begin{aligned}J&=B\exp[-br^2],\\ \Re&=\exp[-\alpha r^2],\\ Q&=\exp[-\beta\rho^2],\\ w&=\exp[-\gamma(r_{56}^2+r_{57}^2+r_{67}^2)],\end{aligned} \quad (45)$$

one has

$$(\Delta E)_{\text{ordinary}}=-(2\lambda^2/3)Bb[(2\alpha\beta\gamma)/(2\alpha\beta\gamma+b(\beta\gamma+\alpha\gamma+2\alpha\beta/9))]^{5/2}. \quad (46)$$

Consider various limiting cases of this result: For $\alpha\to\infty$; $\gamma\to\infty$ (concentrated alpha-group, concentrated triton)

$$(\Delta E)_{\text{ordinary}}=-(2\lambda^2/3)Bb[(2\beta)/(2\beta+b)]^{5/2}. \quad (47)$$

For $\alpha\to\infty$ (concentrated alpha-group)

$$(\Delta E)_{\text{ordinary}}=-(2\lambda^2/3)Bb\times[(2\beta)/(2\beta+b(1+(2/9)\beta/\gamma)]^{5/2}. \quad (48)$$

For $\gamma\to\infty$ (concentrated triton)

$$\begin{aligned}(\Delta E)_{\text{ordinary}}&=-(2\lambda^2/3)Bb\times[(2\alpha\beta)/(2\alpha\beta+b(\beta+\alpha))]^{5/2}\\&=1/9(\Delta E)_{\text{central model He}^5\text{, ordinary terms}}.\end{aligned} \quad (49)$$

Discussion of Exchange Terms for Li^7, Alpha-Triton Model

It will be shown below that the exchange integrals in the limit of small overlapping of the group wave functions have a value proportional to the degree of overlapping. It is of great assistance to be able to eliminate double and triple exchange integrals from the calculations. This may be done by the following reasoning in the case of a small overlap. On a central field model of Li^7 one has the following sequence, where v is a central field function for a particle in the triton:

Ordinary:

$$\int R_1^2R_2^2R_3^2R_4^2v_5^2v_6^2v_7^2d\tau_1\cdots=1,\text{ say.}$$

Single exchange:

$$\int R_1^2R_3^2R_4^2v_5^2v_6^2d\tau_{13456}\int R_2v_2R_7v_7d\tau_{27}=x.$$

Double exchange:

$$\int R_1^2R_4^2v_6^2d\tau_{146}\int R_2v_2R_7v_7d\tau_{27}\times\int R_3v_3R_5v_5d\tau_{35}=x^2.$$

Triple exchange:

$$\int R_1^2 d\tau_1 \int R_4 v_4 R_6 v_6 d\tau_{46} \int R_3 v_3 R_5 v_5 d\tau_{35} \times \int R_2 v_2 R_7 v_7 d\tau_{27} = x^3.$$

If x is small, then x^2 and x^3 may be neglected. On going over now to the case of a group model for the triton, it will be seen that the corresponding double and triple exchange terms are smaller than x^2 and x^3, respectively. Suppose that the contribution x to the single exchange integral comes from a small region of space. It may not be possible for more than one particle in the triton to occupy this region simultaneously, because the center of mass of the triton is restricted in its motion to remain, say, outside of a certain spherical region. In this example the double and triple exchange terms would vanish. These terms may become as large as x^2 and x^3 only if the triton is completely included in the sphere of the alpha-particle.

The general plan of this section is to work out the single exchange terms in a form depending on the overlapping, and certain vector and scalar products. These products will first be estimated roughly in the limit of a very small overlap, and the values obtained in this manner will be the basis for a classification of the terms as large or small. The small terms will then be neglected. The terms which are not neglected will be estimated on a more refined model. Henceforth one assumes

$$(\Psi, H'\Psi^A) \cong I_0 + I_1. \tag{50}$$

Division of Terms into Large and Small Parts

The division proceeds by calculating the averages of various vector and scalar products appearing in I_1, such as $[\mathbf{r}_5 \times \mathbf{r}_2]^2$ and $(\mathbf{r}_7 \cdot \boldsymbol{\rho})$, in the limiting case of small overlap for very simple group wave functions.

The result of the calculation of the averages is that

$$I_1 = -(\lambda^2/108) C \int \mathfrak{R}_1^2 \mathfrak{R}_2^2 \mathfrak{R}_4^2 \mathfrak{R}_3 \mathfrak{R}_5 w(567) \times w(367) Q_\rho Q_\eta [\boldsymbol{\rho} \cdot (10\mathbf{r}_7 - 4\mathbf{r}_3)] d\tau_1 \cdots 7, \tag{51}$$

where several small terms contributing about 1 percent of the result have been neglected. *This is free from any direct assumption about the wave functions.* It need be supposed only that the "small" terms are always negligible and that $J'/r = C$.

Measure of Overlap

The term $(\psi, P_{35}\psi)$ in the normalization integral will be used as a measure of the overlapping. It is supposed that $(\psi, \psi) = 1$.

$$\begin{aligned}(\psi, P_{35}\psi) &= \int (\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_4)^2 \mathfrak{R}_3 \mathfrak{R}_5 w(567) w(367) \times Q_\rho Q_\eta (x_\rho - iy_\rho)(x_\eta + iy_\eta) d\tau \\ &= (2/3) \int (\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_4)^2 \mathfrak{R}_3 \mathfrak{R}_5 w(567) \times w(367) Q_\rho Q_\eta \rho^2 d\tau,\end{aligned} \tag{52}$$

for small overlapping, in which case the whole contribution to the integral comes from $\boldsymbol{\rho} \cong \mathbf{n}$.

Define

$$\Omega = (8d^2/3) \int (\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_4)^2 \mathfrak{R}_3 R_5 w(567) \times w(367) Q_\rho Q_\eta d\tau, \tag{53}$$

which is the value of the above integral for triton radius = alpha-radius = d in the limit of small overlap. It is, however, approximately valid as a measure of overlap for finite overlap—the error is that of replacing $(\boldsymbol{\rho} \cdot \mathbf{n})$ in the integral by the constant $2d^2$.

On this model,

$$I_1 = -7\lambda^2 C\Omega/16, \tag{54}$$

and to the same approximation

$$\begin{aligned}(\Psi, \Psi^A) &= (\psi, \psi) - 3(\psi, P_{35}\psi) \\ &= 1 - 3\Omega,\end{aligned} \tag{55}$$

when we take $(\psi, \psi) = 1$. Thus the complete splitting is

$$\begin{aligned}(\Delta E) = -(\lambda^2/2)[&(\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3 \mathfrak{R}_4)^2 \times J_{71}[2Q_\rho^2 + 4\rho Q'_\rho Q_\rho/3] w^2(567) d\tau \\ &+ 7C\Omega/16]/[\int (\mathfrak{R}_1 \mathfrak{R}_2 \mathfrak{R}_3 \mathfrak{R}_4 \times Q_\rho w(567))^2 d\tau - 3\Omega].\end{aligned} \tag{56}$$

It is instructive to evaluate (ΔE) wholly on the model used for the exchange term:

$$\begin{aligned}(\Delta E) &= -\lambda^2[-C(\psi, \psi) + 7C\Omega/16]/[(\psi, \psi) - 3\Omega] \\ &\cong C\lambda^2(1 + 2.6\Omega).\end{aligned} \tag{57}$$

The exchange term has the same sign as the ordinary term, in agreement with the central model result. The doublet is inverted, since C is negative.

The overlapping can be calculated directly, with Gaussian functions and a product type central field representation of the alpha-group as in (46):

$$\Omega=[12\alpha\beta/(3\alpha^2+6\beta\gamma+6\alpha\gamma+14\alpha\beta/3)]^{\frac{3}{2}}, \quad (58)$$

where several very small terms have been omitted.

The range of the wave function $\mathfrak{R}$ should be corrected for the center of mass motion. This can be estimated from the result for the mean square displacement of the center of mass of the alpha-particle:

$$\langle\zeta^2\rangle_{\mathrm{Av}}=\Big[\int(\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3\mathfrak{R}_4)^2 \times(r_1^2+r_2^2+r_3^2+r_4^2)d\tau_{1\cdots 4}\Big]\Big/\Big[16\int(\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3\mathfrak{R}_4)^2 d\tau_{1\cdots 4}\Big] =3\alpha/16, \quad (59)$$

where ζ is the coordinate of the c.m. of the alpha-particle group. Suppose now α on an independent particle model is given, and one wishes to find the corrected range. The apparent range is $1/\surd\alpha=r_\alpha$; then the corrected range is

$$r_\alpha-(\langle\zeta^2\rangle_{\mathrm{Av}})^{\frac{1}{2}}=(1/\alpha)^{\frac{1}{2}}-(3/16)^{\frac{1}{2}}(1/\alpha)^{\frac{1}{2}} =(1-\sqrt{3}/4)r_\alpha.$$

The overlap integral $(\psi, P_{35}\psi)=\Omega$ can be approximately related to the periods of motion of the particles in the alpha-group and the center of mass of the triton, considering the triton group to be very concentrated. By utilizing the uncertainty principle, we have

$$\begin{aligned}\Omega &\cong \text{range particles in alpha-group}/\text{range c.m. of triton}\\ &\cong \text{momentum c.m. of triton}/\text{momentum particles in alpha-group}\\ &\cong \surd\ \text{period particles in alpha-group}/\surd\ \text{period c.m. of triton}. \quad (60)\end{aligned}$$

If the alpha-group is considered to be very concentrated,

$$\Omega=\surd\ \text{period c.m. of alpha-group}/\surd\ \text{period particles in triton}. \quad (61)$$

Similar estimates can be made by using the W. K. B. approximation as a solution of the radial wave equations for the various particles and groups, but no exact connection can be made between the overlap and the periods of motion, because the spatial characteristics of wave functions such as the presence of "tails" affect the overlap integrals.

The splitting for Li^7 on a true alpha-particle model in which only relative coordinates appear has not been discussed because of the great amount of labor involved in treating such a model. An investigation of the geometrical conditions for the vanishing of the exchange terms has been carried out, however, in a manner similar to that employed on the same question for He^5. It turns out that for non-overlapping groups the single and double exchange integrals I_1 and I_2 vanish, but the triple exchange integral I_3 does not vanish until the distance between the c.m.'s of the two groups is at least 7/3 the radius of the alpha-group.

VI. CONCLUSIONS

1. The contribution of the exchange terms relative to ordinary to the splitting in He^5 is greater by a factor of the order of two or three on the alpha-model than on the central field model. The sign and magnitude of the total splitting are of about the same order on the two models.

2. The "deuteron-grouping" alpha-function gives for He^5 a splitting within a few percent of that for the ordinary alpha-function.

3. The exchange terms for Li^7 are approximately proportional to the degree of overlapping of the alpha- and triton groups. The exchange splitting has the same sign as the ordinary splitting.

4. Energy splitting estimates on the alpha-model in which the exchange terms are neglected should give results reliable to an order of magnitude at least so far as this model is concerned. On reasonable assumptions the value of the exchange terms is of the order of one-half of that of the ordinary terms.

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